Solution to Assignment 10

Supplementary Exercise

1. (a) Show that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n} + \int_0^x \frac{(-t)^n}{1+t}dt \ .$$

Suggestion: Think about

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + \frac{(-x)^n}{1+x} \ .$$

(b) Show that

$$\left|\log(1+x) - \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1}\frac{x^n}{n}\right)\right| \le \frac{x^{n+1}}{n+1} \ .$$

Solution. (a) follows from a direct integration. The second inequality follows from the first inequality after noting

$$\left|\int_0^x \frac{(-t)^n}{1+t}dt\right| \le \int_0^x t^n dt = \frac{x^{n+1}}{n+1} \ .$$

2. This exercise suggests an alternative way to define the logarithmic and exponential functions. Define nog : $(0,\infty) \to \mathbb{R}$ by

$$\text{nog}(x) = \int_1^x \frac{1}{t}dt.$$

(a) nog(x) is strictly increasing, concave, and tends to ∞ and $-\infty$ as $x \to \infty$ and 0 respectively.

(b) nog(xy) = nog(x) + nog(y).

(c) Define $e(x)$ to be the inverse function of nog. Show that it coincides with $E(x)$.

Note: f is concave means $-f$ is convex. You cannot assume $\log x$ has been defined.

Solution.

(a) By fundamental theorem of calculus, nog is differentiable and $(\text{nog } x)' = \frac{1}{x} > 0$. Hence it is strictly increasing. Moreover, $(\text{nog } x)'' = -\frac{1}{x^2} < 0$ hence it is strictly concave. Next we observe $\forall\ x \ge 2$, $\exists\ n_x \in \mathbb{R}$ s.t. $n_x - 1 \le x < n_x$. Then

$$\begin{aligned}
\text{nog } x \ge \text{nog}(n_x - 1) &= \int_1^{n_x-1} \frac{1}{t} \\
&= \sum_{k=2}^{n_x-1} \int_{k-1}^{k} \frac{1}{t} \ge \sum_{k=2}^{n_x-1} \int_{k-1}^{k} \frac{1}{k} \\
&= \sum_{k=2}^{n_x-1} \frac{1}{k} \ .
\end{aligned}$$

Letting $x \to \infty$, $n_x \to \infty$, hence $\lim_{x\to\infty} \text{nog } x \geq \sum_{k=2}^{\infty} \frac{1}{k} = \infty$.
Next, by the change of variables $s = 1/t$,

$$\text{nog } x = \int_1^x \frac{dt}{t} = \int_{1/x}^1 \frac{ds}{s} \to -\infty \ ,$$

as $x \to 0$.

(b)

$$\begin{aligned}\text{nog } xy &= \int_1^{xy} \frac{1}{t}\, dt \\ &= \int_1^x \frac{1}{t}\, dt + \int_x^{xy} \frac{1}{t}\, dt \\ &= \int_1^x \frac{1}{t}\, dt + \int_x^{xy} \frac{1}{xt}\, d(xt) \ ,\end{aligned}$$

since $x > 0$. It is equal to

$$\int_1^x \frac{1}{t}\, dt + \int_1^y \frac{1}{u}\, du = \text{nog } x + \text{nog } y \ .$$

(c) From (a), nog is strictly increasing hence one-to-one, its inverse function $e(x)$ is well defined.

$$e'(x) = \frac{1}{(\text{nog})'(e(x))} = \frac{1}{1/e(x)} = e(x) \quad \forall\, x \in \mathbb{R} \ ,$$

and $e(0) = 1$ since nog $(1) = 0$. By uniqueness, $e(x)$ coincides with $E(x)$.
Note. This approach has a drawback, namely, it is not feasible for generalization.

3. Show that there is a unique solution $c(x), x \in \mathbb{R}$, to the problem

$$f'' = f, \quad f(0) = 1, \ f'(0) = 0.$$

(a) Letting $s(x) \equiv c'(x)$, show that s satisfies the same equation as c but now $s(0) = 0, \ s'(0) = 1$.

(b) Establish the identities, for all x,

$$c^2(x) - s^2(x) = 1,$$

and

$$c(x + y) = c(x)c(y) + s(x)s(y).$$

(c) Express c and s as linear combinations of e^x and e^{-x}. (c and s are called the hyperbolic cosine and sine functions respectively. The standard notations are $\cosh x$ and $\sinh x$. Similarly one can define other hyperbolic trigonometric functions such as $\tanh x$ and $\coth x$.)

Solution. They are parallel to the case of E. We only consider the uniqueness issue. As in the case for the exponential function, it suffices to show if both g satisfy $g'' = g$, $g(x_0) = g'(x_0) = 0$ at some x_0, then $g \equiv 0$. Well, it is a direct check that g satisfies the integral equation

$$g(x) = \int_{x_0}^{x} \int_{x_0}^{t} g(z)dz \ .$$

We claim $g \equiv 0$ on $[x_0 - 1, x_0 + 1]$. For, let $M = |g(x_1)|$ be the max of $|g|$ on this interval. We have

$$M = |g(x_1)| \le \left| \int_{x_0}^{x} \int_{x_0}^{t} g(z)dz \right| \le M \int_{x_0}^{x} \int_{x_0}^{t} dz = M \frac{(x - x_0)^2}{2} \le \frac{M}{2} \ ,$$

which forces $M = 0$.

Remark. The functions c and s are actually the hyperbolic cosine and sine functions given respectively by

$$\cosh x = \frac{e^x + e^{-x}}{2} \ , \quad \sinh x = \frac{e^x - e^{-x}}{2} \ .$$